Nonlinear Diffusion and its applications

Javier Chico Vázquez

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Contents

1 Introduction

In the present text we will introduce the concept of nonlinear diffusion, its properties and applications. The problem will be introduced by deriving the porous media equation from basic physical laws, followed by finding a similarity solution and a discussion of some of the most relevant properties of this solution. Finally, the results are connected to lubrication theory, a rich area of fluid dynamics with profound applications.

We present results first derived by Zeldovich, Kompaneetsm and Barenblatt (ZKB) [1]. Currently, much focus is placed not just in nonlinear diffusion but also in diffusion in non local media [2], which is a related topic. Furthermore, both flow in porous media and lubrication theory are areas of active research [3,4].

However much of the literature does not pay attention to the closed form formulas for the normalisation constants that appear in ZKB solutions [2]. Hence, particular attention was placed in finding such closed formulas in the one dimensional case. Moreover, a new phenomenon is explored where the existence local minima in the front position (as a function of the nonlinearity) will depend on a critical time.

This paper is organized in the following way. First the nonlinear diffusion equation is derived as the porous media equation in Section 2. In the next section (3) the Zeldovich-Kompaneets-Barenblatt (ZKB) similarity solution is derived. Section 4 covers the fundamental properties of the ZKB solutions, and discusses the non-monotonic behaviour of the front propagation. Finally, we discuss possible connections to other areas of fluid mechanics, such as lubrication theory in Section 5.

Nonlinear diffusion follows very naturally as an extension to the linear diffusion equation studied in lectures. Moreover, it displays exotic behaviour that will be discussed in detail later, and can explain more physical and biological phenomena than its linear counterpart.

2 Derivation from flow in Porous media

We will derive the non linear diffusion equation from an application in physics, much like we did with the Laplace equation (e.g. Gravity), heat equation (e.g. heat transport) and wave equation (e.g. tension on a string). We can imagine gas flowing through a porous media, with porosity $f \in [0,1]$. Porosity is defined as the fraction of the media which is "empty", meaning the open space where gas can flow. As usual, let ρ denote the gas density and $\mathbf{v}(\mathbf{x})$ the velocity of the gas. The first equation we know must hold is conservation of mass, which now takes the form

$$f\rho_t + \nabla \cdot (\rho \mathbf{v}) = 0$$

where we have accounted for the porosity of the media and as usual we are using $\mathbf{q} = \rho\mathbf{v}$ as mass flux. Much like in the case of the linear heat equation, we now need a constitutive relationship which relatives the flux to ρ and its derivatives. In that case we used Fourier's law, $\mathbf{q} \sim -\nabla\rho$. However, now we will use Darcy's law [2]

$$\mathbf{v} = -\frac{\kappa}{\mu}\nabla p$$

where p is the pressure, μ is the viscosity and κ is the permeability. Finally, as we have introduced the pressure as a new variable, we must complete our set of equations with an equation of state for the gas, taken for now as an adiabatic equation of state

$$\rho = \rho_0 p^\gamma$$

Using these equations we can get a single PDE for ρ

$$p = \left(\frac{\rho}{\rho_0}\right)^{(1/\gamma)}$$

then

$$\mathbf{v} = -\frac{\kappa}{\mu}\nabla p = -\frac{\kappa}{\mu}\nabla\left(\frac{\rho}{\rho_0}\right)^{(1/\gamma)}$$

Now, $\nabla\rho^a = a\rho^{a-1}\nabla\rho$, so this gives us

$$\mathbf{v} = -\frac{\kappa}{\mu}\nabla\left(\frac{\rho}{\rho_0}\right)^{(1/\gamma)} = -\frac{\kappa}{\gamma\mu\rho_0^{1/\gamma}}\rho^{1/\gamma-1}\nabla\rho$$

Going back to the conservation of mass equation:

$$f\rho_t + \nabla \cdot (\rho \mathbf{v}) = 0$$

so

$$f\rho_t - \nabla\cdot\left(\rho\frac{\kappa}{\gamma\mu\rho_0^{1/\gamma}}\rho^{1/\gamma-1}\nabla\rho\right) = 0 \implies f\rho_t - \frac{\kappa}{\gamma\mu\rho_0^{1/\gamma}}\nabla\cdot(\rho^{1/\gamma}\nabla\rho) = 0$$

$$f\rho_t - \frac{\kappa}{\gamma\mu\rho_0^{1/\gamma}}\frac{1}{1/\gamma+1}\nabla^2(\rho^{1/\gamma+1}) = 0$$

So taking $m = \frac{1}{\gamma} + 1$ and $c = -\frac{\kappa}{f\gamma\mu\rho_0^{1/\gamma}}\frac{1}{1/\gamma+1}$ we get the equation we wanted

$$\rho_t = c\nabla^2\rho^m$$

Finally, re-scaling time by c we get a non dimensional equation

$$\rho_t = \nabla^2\rho^m$$

Now that we have derived the equation we can move on and start to discuss the properties of its solutions

3 ZKB solutions. Derivation of the solution from a similarity solution

In this section we will assume an unbounded domain. We will assume the initial distribution has all its mass at $x = 0$, so that

$$\rho(x, 0) = \delta(x)$$

and far field boundary conditions

$$\lim_{|x|\to\pm\infty} \rho(x, t) = 0$$

We will enforce conservation of mass by forcing

$$1 = \int_{-\infty}^{\infty} \rho(x, t)dx$$

Before going into the general case, it is important to remark that for $m = 1$ we just get the standard linear diffusion equation, which everyone is familiar with, as well as its solutions given the above initial conditions.

$$\partial_t\rho = \nabla^2\rho$$

In D space dimensions and when $\rho(x, 0) = \delta(x)$ we recover the fundamental solution/kernel, namely

$$\rho^*(x, t) = \left(\frac{1}{4\pi t}\right)^{D/2} \exp\left(-\frac{|x|^2}{4t}\right)$$

And we can use this solution to get the solution for an arbitrary initial condition $\rho(x, 0) = \phi(x)$, using

$$\rho(x, t) = \int \rho^*(x - y, t)\phi(y)dy$$

However, with this solution information travels infinitely fast! This is because for any $t > 0$ $u(x, t) > 0$ for all x, regardless of how far away from the origin. To get a solution where this is not the case we return to the nonlinear diffusion equation and use $m \neq 1$

$$\partial_t\rho = \nabla^2[\rho^m]$$

We will see a similarity solution

$$\rho(x, t) = t^{-\alpha}U(xt^{-\beta}) = t^{-\alpha}U(\xi)$$

with $\xi = xt^{-\beta}$

3.1 One dimensional domain

Assume for now the domain is one dimensional, so that $\nabla^2 = \partial_{xx}$. Conservation of mass can be enforced with

$$1 = \int_{-\infty}^{\infty} \rho(x,t)dx = \int_{-\infty}^{\infty} t^{-\alpha} U(xt^{-\beta})dx = t^{-\alpha} \int_{-\infty}^{\infty} U(\xi) t^{\beta} d\xi$$

we see we need $\alpha = \beta$ from conservation of mass (this integral is constant so it can not depend on time). Hence

$$\rho(x,t) = t^{-\alpha} U(xt^{-\alpha})$$

We can now substitute this into the PDE to see what value of α we need. We can compute the derivatives in the PDE

$$\rho_t = -\alpha t^{-\alpha-1}(U + \xi U')$$
$$(\rho^m)_{xx} = t^{-\alpha m} t^{-2\alpha} (U^m)''$$

and now substitute to get

$$-\alpha t^{-\alpha-1}(U + \xi U') = t^{-\alpha m} t^{-2\alpha} (U^m)''$$

To eliminate the time dependence and get an ODE in ξ only we need to impose

$$-\alpha - 1 = -\alpha m - 2\alpha$$

thus to have a similarity solution we have to pick

$$\alpha = \frac{1}{m+1}$$

and we get the following ODE for U:

$$(m+1)(U^m)'' + \xi U' + U = 0$$

We can now use a trick to write this as

$$\frac{d}{d\xi}[(m+1)(U^m)' + \xi U] = 0$$

which we can integrate easily yo

$$(m+1)(U^m)' + \xi U = C$$

We can figure out the value of C using the positive far-field boundary condition, which requires $C = 0$. Thus

$$(m+1)(U^m)' = -\xi U$$

We can expand the left hand side to get

$$(m+1)mU^{m-1}U' = -\xi U$$

and thus

$$(m+1)mU^{m-2}U' = -\xi$$

then, as $m \neq 1$

$$\frac{(m+1)m}{m-1}(U^{m-1})' = -\xi \quad \Longrightarrow \quad C_m (U^{m-1})' = -\xi$$

we can integrate this to get

$$C_m U^{m-1} = -\frac{\xi^2}{2} + A$$

And now we can solve for U

$$U(\xi) = [A - B_m \xi^2]^{1/(m-1)}$$

undoing the similarity transformation we get ρ

$$\rho(x,t) = t^{-\alpha} \left[A - B_m \frac{x^2}{t^{2\alpha}} \right]^{\frac{1}{m-1}}$$

when $x^2 \leq At^{2\alpha}/B_m$ and 0 otherwise. Here we have $B_m = \frac{m-1}{2m(m+1)}$ and A is a normalization constant to ensure mass is conserved. In the literature A is often not computed, but we can compute A by forcing

$$1 = \int_{-\sqrt{A/B_m}t^\alpha}^{\sqrt{A/B_m}t^\alpha} t^{-\alpha} \left[A - B_m \frac{x^2}{t^{2\alpha}} \right]^{\frac{1}{m-1}} dx = \int_{-\sqrt{A/B_m}}^{\sqrt{A/B_m}} \left[A - B_m \xi^2 \right]^{\frac{1}{m-1}} d\xi$$

Now taking $\xi = \sqrt{A/B_m}\eta$ we get

$$1 = \sqrt{\frac{A}{B_m}} A^{1/(m-1)} \int_{-1}^{1} (1 - \eta^2)^{1/(m-1)} d\eta$$

We can integrate this using tabulated integrals related to the Beta distribution

$$\int_{-1}^{1} (1 - \eta^2)^{1/(m-1)} d\eta = \frac{\sqrt{\pi}\Gamma(\frac{m}{m-1})}{\Gamma(3/2 + \frac{1}{m-1})}$$

thus we can express A analytically with

$$A = A_m = \left[\sqrt{\frac{B_m}{\pi}} \frac{\Gamma(3/2 + \frac{1}{m-1})}{\Gamma(\frac{m}{m-1})} \right]^{\frac{2(m-1)}{m+1}}$$

3.2 Beyond one dimension

We can use the same idea to build a solution for an arbitrary dimension, much like in the case of the linear diffusion equation. Hence, suppose we have the following problem in $\mathbb{R}^N$ (slightly modified from the problem for $\mathbb{R}$) [2,5]

$$u_t - \nabla \cdot (u^\sigma \nabla u) = 0$$

subject to

$$\int_{\mathbb{R}^N} u(x,t)(dx)^N = 1 \quad \text{and} \quad u(x,t) \geq 0$$

and with initial conditions

$$u(x,0) = \delta(x)$$

Here $\sigma = m - 1$ with m as in the one-dimensional case. We can proceed as before case and seek a similarity solution of the form

$$u(x,t) = t^\alpha \theta(\xi)$$

with $\xi = xt^\alpha$. However, unlike before, the values of α and β will depend on the dimension N.In particular, the mass conservation condition will imply that

$$\alpha + N\beta = 0$$

And upon substitution to the governing equation we have another equation that reads $\sigma\alpha - 2\beta = -1$ We can solve the above to get:

$$\alpha = -\frac{N}{2 + N\sigma} \quad \beta = \frac{1}{2 + N\sigma}$$

In particular, when we substitute $N=1$ and $\sigma = m-1$ we recover the same as in the one dimensional case. We thus now have that

$$\nabla \cdot (\theta^\sigma \nabla \theta) + \frac{N}{2+N\sigma}\theta + \frac{1}{2+N\sigma}\xi \cdot \nabla\theta = 0$$

We can now argue that the solution will be radially symmetrical, so that $\theta = \theta(\eta)$ where $\eta = |\xi|$. After some algebra involving the divergence in spherical coordinates this will give us

$$\frac{1}{\eta^{N-1}}(\eta^{N-1}\theta^\sigma\theta')' + \frac{N}{2+N\sigma}\theta + \frac{1}{2+N\sigma}\eta\theta' = 0$$

which can be written as

$$(\eta^{N-1}\theta^\sigma\theta')' + \frac{1}{2+N\sigma}(\eta^N\theta)' = 0$$

We can integrate the above directly, and as in the one dimensional case argue that the integration must be zero as $\theta'(0) = 0$. Hence,

$$\eta^{N-1}\theta^\sigma\theta' + \frac{1}{2+N\sigma}\eta^N\theta = 0$$

Now this is re-expressed as

$$\theta^{\sigma-1}\theta' = -\frac{1}{2+N\sigma}\eta \quad\Longrightarrow\quad \frac{1}{\sigma}\theta^\sigma = \frac{1}{2(2+N\sigma)}(\eta_0^2 - \eta^2)$$

when $0 \le \eta \le \eta_0$. The non-negativy constraint ensures that the solution is identically zero outside of this domain. Finally we can solve for θ and invert the similarity transformation

$$\theta(\eta) = \left(\frac{1}{2(2+N\sigma)}(\eta_0^2 - \eta^2)\right)^{1/\sigma} \quad\Longrightarrow\quad u(x,t) = t^{-\frac{N}{2+N\sigma}}\left(\frac{1}{2(2+N\sigma)}\left(\eta_0^2 - \left(\frac{|x|}{t^{\frac{1}{2+N\sigma}}}\right)^2\right)\right)^{1/\sigma}$$

4 Properties of the ZKB solution

In this section we will focus on the one dimensional case for convenience.

4.1 Compact Support

The most interesting fact about the ZKB solution is that the solution is zero from a finite value of x onwards, in particular we have a true front at

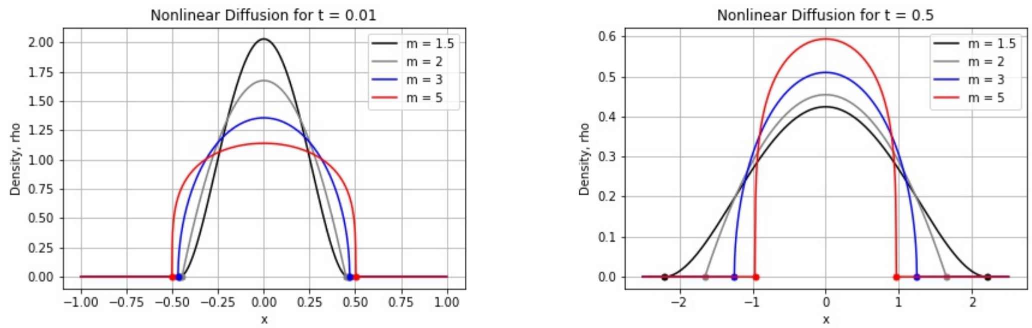
$$x_f = r(t) = \pm\sqrt{A/B_m}t^\alpha$$

By true, we mean that the solution is zero outside the inside of that radius. We can visualize this for several values of m at a fixed time in Figure 1a and Figure 1b.

4.2 Front propagation velocity

From the formula for the front position we can derive the front speed by taking a time derivative:

$$r'(t) = \alpha\sqrt{A/B_m}t^{\alpha-1}$$



(a) The solution at $t = 0.01$ for several values of m (b) The solution at $t = 0.5$ for several values of m

Figure 1: ZKB solutions at several times for a selection of nonlienarity orders. Note that value of m which has the maximum value and the biggest front support interval changes between the two times.

4.3 What nonlinearity degree m has the smallest radius?

When we study the function $r_m(t)$ for a fixed time we observe an interesting behaviour. For small times, the front position for low m seems smaller than that for high m, but for big times it looks like r_m is a monotonically decreasing function of m. This can be seen in Figure 2

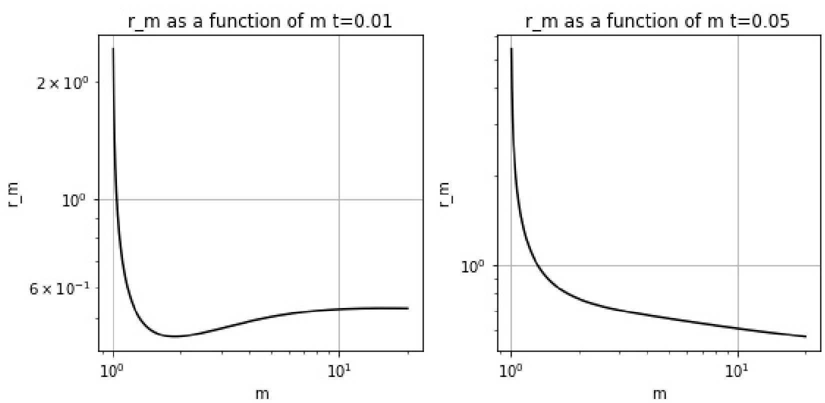


Figure 2: Front radius as a function of the nonlinear exponent m at different times. We can see a qualitative change in the plots: there is a clear local minimum for small times which does not appear to be present for large times.

We can explore this phenomena further, and try to get an analytical estimate for this critical time for which the minima disappears. If there is a minimum present, it will be a solution to

$$\frac{\partial r_m(t)}{\partial m} = 0$$

However, recall that

$$r_m(t) = \sqrt{A/B_m} t^{\alpha} \quad , \quad A = A_m = \left[\sqrt{\frac{B_m}{\pi}} \frac{\Gamma(3/2 + \frac{1}{m-1})}{\Gamma(\frac{m}{m-1})} \right]^{\frac{2(m-1)}{m+1}} \quad \text{and} \quad \alpha = \alpha(m) = \frac{1}{m+1}$$

We can split the above expression by defining

$$\pi(m) = \pi^{-\frac{m-1}{m+1}} \quad \text{and} \quad \gamma(m) = \left[\frac{\Gamma(3/2 + \frac{1}{m-1})}{\Gamma(\frac{m}{m-1})} \right]^{\frac{2(m-1)}{m+1}}$$

So we have

$$A_m = \pi(m)\gamma(m) B_m^{\frac{m-1}{m+1}}$$

so that

$$r_m(t) = \sqrt{A/B_m} t^\alpha = \sqrt{\pi(m)\gamma(m) B_m^{\frac{m-1}{m+1}} / B_m t^\alpha} = \sqrt{\pi(m)\gamma(m)} (t/B_m)^\alpha$$

Numerical results indicate that $\gamma(m)$ varies slowly with m, at least compared to $\pi(m)$ and B_m, so as a first approximation it is reasonable to assume that it is indeed constant with respect to m. Hence, we can discard that term when computing the derivative

$$\partial_m r_m(t) \approx \partial_m[\sqrt{\pi(m)}(t/B_m)^\alpha] = \partial_m[\sqrt{\pi(m)}](t/B_m)^\alpha + \sqrt{\pi(m)}\partial_m[(t/B_m)^\alpha] = 0$$

We can now focus on each of the individual derivatives

$$\partial_m[\sqrt{\pi(m)}] = \frac{1}{2\sqrt{\pi(m)}} \partial_m \pi(m) \quad \partial_m \pi(m) = -\frac{2\pi(m)\log(\pi)}{(m+1)^2}$$

and

$$\partial_m[(t/B_m)^\alpha] = \left(\partial_m \alpha \log\left(\frac{t}{B_m}\right) + \alpha B_m \partial_m \left(\frac{1}{B_m}\right) \right) (t/B_m)^\alpha$$

Now focusing on each of the individual derivatives

$$\partial_m \alpha = -\frac{1}{(m+1)^2} \quad \text{and} \quad B_m \partial_m \left(\frac{1}{B_m}\right) = -\frac{1}{B_m}\partial_m B_m = -\frac{1}{B_m}\left(\frac{-m^2+2m+1}{2m^2(m+1)^2}\right)$$

We can now go back to the original expression for the derivative and substitute

$$\partial_m[\sqrt{\pi(m)}(t/B_m)^\alpha] = \left(-\frac{\sqrt{\pi(m)}\log \pi}{(m+1)^2} + \sqrt{\pi(m)} \left[-\frac{\log(t/B_m)}{(m+1)^2} - \frac{\alpha}{B_m}\left(\frac{-m^2+2m+1}{2m^2(m+1)^2}\right) \right] \right) (t/B_m)^\alpha = 0$$

We can simplify the above to

$$-\log \pi - \log(t/B_m) - \frac{\alpha}{B_m}\left(\frac{-m^2+2m+1}{2m^2}\right) = 0$$

and when we substitute the values for α and B_m we get

$$-\log \pi - \log t + \log \frac{m-1}{2m(m+1)} - \left(\frac{-m^2+2m+1}{m(m-1)}\right) = 0$$

So we are now interested in the maximum value of t for which this equation has a solution $m^* > 1$. We can approach this question by writing the above as

$$-\log \pi + \log \frac{m-1}{2m(m+1)} - \left(\frac{-m^2+2m+1}{m(m-1)}\right) = \log t$$

This equation will have a solution if and only if the minimum value of the right hand side is less than $\log t$. Hence, we can compute the maximum value of the right hand side, which we denote as $\psi(m)$. To do this we take another derivative and solve for m. We get

$$\partial_m(\psi(m)) = 0 = \partial_m \left[\log \frac{m-1}{2m(m+1)} - \left(\frac{-m^2+2m+1}{m(m-1)}\right) \right] = \frac{m^4 - 4m^3 - 2m^2 + 1}{m^2(m^3 - m^2 - m - 1)} = 0$$

So we are interested in the roots of the quartic polynomial

$$m^4 - 4m^3 - 2m^2 + 1 = 0$$

The only root greater than one is $m^* = 4.4391\ldots$. This gives an approximate critical time of $t_c \approx 0.043164$.

4.4 Maximum value

In the linear case, the maximum is always located at $x = 0$, and the maximum value decays as

$$u_{max} \sim \frac{1}{\sqrt{t}}$$

As the maximum value is always at $x = 0$, we can find the maximum value for the nonlinear case by substituting $x = 0$. It turns out to be

$$\rho_{max}(t) = t^{-\alpha}(A)^{\frac{1}{m-1}} = t^{-\frac{1}{m+1}}(A)^{\frac{1}{m-1}} \sim t^{-\frac{1}{m+1}}$$

So in particular we see the solution matches if we substitute the linear value $m = 1$, which is unexpected as normally $m = 1$ is a singular/degenerate parameter of the nonlinear problem. We can see this by the appearance of $\frac{1}{m-1}$ terms which are undefined for $m = 1$

5 Application and similarity to lubrication theory

The lubrication limit is a simplification of the Navier-Stokes equations which is valid when one of the spatial dimensions is much smaller than the others. In two spatial dimensions, if we are dealing with a thin film, characterised by length L in the x-direction and H in the y-direction, with $H/L << 1$ (see [4]), and low Reynolds number, we can reduced the incompressible Navier-Stokes equations to [6, 4,7]

$$\frac{\partial p}{\partial x} = \mu \frac{\partial^2 u}{\partial y^2}$$

Where μ is the dynamic viscosity, u is the vertical component of the velocity and p is the pressure. We also have no slip boundary conditions on the bottom of the film, $v(0) = 0$, and at the top (denote by $y = h(x, t)$) we have $v(h) = \frac{\partial h}{\partial t}$. Furthermore we have no slip for the horizontal component u in every boundary. If we assume the pressure varies slowly along y we can integrate the above to

$$u(y) = \frac{1}{2\mu}\frac{\partial p}{\partial x} y(y - h)$$

where the integration constants have been chosen so that the boundary conditions are met. We can now integrate the conservation of mass ($\nabla \cdot \mathbf{u} = 0$) with respect to y to obtain

$$\frac{\partial}{\partial x}\int_0^h u dy + \frac{\partial h}{\partial t} = 0$$

But $\int_0^h u dy$ is just the flux Q, which can be calculated to be

$$Q = -\frac{1}{12\mu}\frac{\partial p}{\partial x} h^3$$

So that we get the Reynolds equations

$$\frac{\partial}{\partial x}\left(-\frac{1}{12\mu}\frac{\partial p}{\partial x} h^3\right) + \frac{\partial h}{\partial t} = 0$$

To close the problem we need to find p as function of h. If only gravity and surface tension are important, we can model $p = p(h)$ by [6, 8]

$$p = -\gamma \nabla^2 h + \rho g h$$

where γ is the surface tension. Upon substitution,

$$\frac{\partial}{\partial x}\left(-\frac{1}{12\mu} h^3 \frac{\partial}{\partial x}(-\gamma \nabla^2 h + \rho g h)\right) + \frac{\partial h}{\partial t} = 0$$

Hence we see that when surface tension is small compared to effects due to gravity the Reynolds equation reduces to the porous media equation, for which we have solution from the previous sections, and has been explored in detail in this context, see for instance Holm et. al. [3]. In particular, we have $\sigma = 3$ or equivalently $m = 4$

$$\frac{\partial}{\partial x}\left(-\frac{\rho g}{12\mu}h^3\frac{\partial}{\partial x}h\right)+\frac{\partial h}{\partial t}=0 \Longleftrightarrow \frac{\partial h}{\partial t}=\frac{\rho g}{12\mu}\frac{\partial}{\partial x}\left(h^3\frac{\partial h}{\partial x}\right)$$

5.1 Non-negligble surface tension and connection to pattern formation

If the surface tension is comparable to gravity effects then we can linearize around some average value of h, say h_0, and write

$$h = h_0 + \varepsilon h_1(x,t)$$

which to first order gies us the lienar equations,

$$\frac{\partial h_1}{\partial t}=\frac{h_0^3}{12\mu}\left(-\gamma\frac{\partial^4 h_1}{\partial x^4}\right)+\rho g\frac{\partial^2 h_1}{\partial x^2}$$

We can find the linear dispersion relation by trying an ansatz $h_1 = e^{\sigma t + ikx}$ which gives us [4,9]

$$\sigma(k)=\frac{h_0^3}{12\mu}[-\gamma k^4+\rho g k^2]$$

thus we see that the system will likely develop patterns when $\sigma > 0$. Furthermore, the wave number that we will observe experimentally is given by $\mathrm{argmax}_k\sigma(k)$, the most unstable node, which turns out to be

$$k^* = \sqrt{\frac{\rho g}{2\gamma}}$$

and so the observed wavelength is

$$\lambda^* = \frac{2\pi}{k^*} = 2\pi\sqrt{\frac{2\gamma}{\rho g}}$$

6 Conclusion and further work

All in all the nonlinear diffusion equation was derived from two different physical contexts. The classical ZKB solution was obtained and several of its properties were extensively studied. In particular, a closed form solution for the normalisation constant was found, and furthermore a previously undescribed phenomena relating the front position with the order of the nonlinearity was analysed in depth. It was observed that there is a bifurcation at a critical time below which a local minimum exists. However, beyond this critical time the front position is a monotonically decreasing function of the order of the nonlinearity, m.

For the lubrication theory application we further explored the role of nonlinear diffusion in pattern formation, alongside other important aspects from lectures, most notably dispersion.

In the future, it would be interesting to seek a better approximation to the critical time for the existence of a local minimum of the front radius, and further study the role nonlinear diffusion plays in pattern formation problems.

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